Inflow-Outflow Boundary Conditions for Two-Dimensional Acoustic Waves in Channels with Flow

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An analysis of the number and form of the required inflow-outflow boundary conditions for the full two-dimensional time-dependent nonlinear acoustic system in subsonic mean flow is performed. The analysis is performed in a rectangular channel with rigid walls and is extendable to more complicated geometries and wall conditions. The explicit predictor-corrector method of MacCormack is used to integrate the acoustic system coupled with the derived boundary conditions. The methodology is tested on both uniform and sheared mean flows with plane and nonplanar sources. Results show that the acoustic system requires three physical boundary conditions on the inflow and one on the outflow boundary. The most natural choice for the inflow boundary conditions is judged to be a specification of the vorticity, the normal acoustic impedance, and a pressure gradient-density gradient relationship normal to the boundary. Specification of the acoustic pressure at the outflow boundary along with these inflow boundary conditions is found to give consistent reliable results. A set of boundary conditions developed earlier by K. W. Thompson, which were intended to be nonreflecting, is tested using the current method and is shown to yield unstable results for nonplanar acoustic waves.

Introduction

A SOLUTION to the equations of acoustics in viscous flow is necessary to understand the interaction of sound and boundary-layer flows. Exact solutions to these equations in their most general nonlinear form are not available, and no exact solutions are likely to be found in the foreseeable future. Recently, there has emerged considerable interest in solving these equations numerically, as an initial-boundary-value problem.[1-5] This has the potential of coping readily with periodic, nonperiodic, or transient acoustic signals and automatically takes into account the nonlinear interharmonic interactions of the acoustic waves in the flow.

As this work has progressed, difficulties with the inflow-outflow boundary conditions have become evident. In nearly all unsteady flows, proper specification of these boundary conditions remains a continuing problem.[1-5,11,13] Researchers in acoustics have avoided this problem by seeking linearized solutions to the acoustic equations in the form of axially propagating waves or by making assumptions that reduce the system to a single linear differential equation.[6-10] In fact, the initial-boundary-value problem governing the nonlinear acoustic system seems to have never been formally analyzed to determine the number or mathematical form of the boundary conditions required for its solution to be well posed.

In this paper, an analysis of the inflow-outflow boundary conditions for the full two-dimensional time-dependent nonlinear acoustic system in a subsonic mean flow is performed using a method originally developed for fluid dynamics.[11] The derived boundary conditions are then examined from a computational standpoint to determine if they reproduce known solutions available for outgoing waves in a parallel flow. The explicit predictor-corrector method of MacCormack is used to numerically integrate the system consisting of the differential equations interior to the domain coupled with the boundary conditions.

Equations Governing the Acoustic System

Figure 1 illustrates the computational domain and coordinate system to be used in the study. The flowing fluid is considered to be subsonic, laminar, viscous, compressible, without mass diffusion and finite-rate chemical reaction, and from left to right. The inflow and outflow boundaries are located at $x = 0$ and $x = L$, respectively, as shown in the figure, and the upper and lower boundaries are considered to be rigid and nonconducting. It is assumed that for all times $t < 0$, the flow in the channel consists only of a known steady flow. However, for times greater than zero, an acoustic pressure source $p_s(y,t)$ is prescribed in addition to the steady flow at the outflow boundary. As is common in acoustics, the source pressure is assumed known for all $t > 0$ at the outflow boundary in the presence of the steady flow. (It is the purpose of this paper to determine the number and form of the inflow-

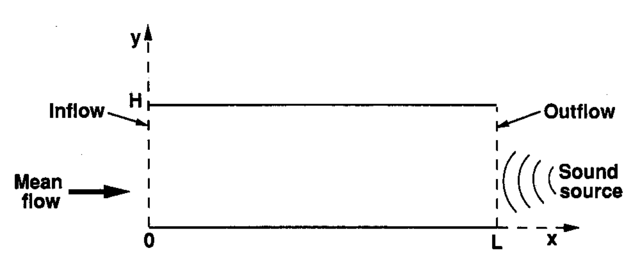

Fig. 1 Rectangular channel and coordinate system.

Presented as Paper 89-1041 at the AIAA 12th Aeroacoustic Conference, San Antonio, TX, April 10–12, 1989; received Jan. 11, 1990; revision received May 30, 1990; accepted for publication Sept. 7, 1990.

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outflow boundary conditions for the full nonlinear acoustic system in the presence of the flowing fluid.)

To begin, the fundamental equations governing the fluid flow are the two-dimensional Navier-Stokes equations; written in conservative form, they are

$$\frac{\partial\{U\}}{\partial t}+\frac{\partial\{H\}}{\partial x}+\frac{\partial\{G\}}{\partial y}=\{0\} \tag{1}$$

The vector components in Eq. (1) are given in virtually any fluid dynamics text (for example, see Ref. 13) and are not written explicitly. The unknown variables in $\{U\}$ are the fluid density ρ, axial component of velocity u, transverse component of velocity v, fluid temperature T, and fluid pressure p. To derive the equations governing the acoustic system in flow, the unknown fluid variables are decomposed into a mean or steady flow and an acoustic perturbation component

$$\{\Phi\}=\{\Phi_0\}+\{\tilde{\Phi}\} \tag{2a}$$

$$\{\Phi\}=\begin{Bmatrix}\rho\\u\\v\\p\end{Bmatrix},\quad \{\Phi_0\}=\begin{Bmatrix}\rho_0\\u_0\\v_0\\p_0\end{Bmatrix},\quad \{\tilde{\Phi}\}=\begin{Bmatrix}\tilde{\rho}\\\tilde{u}\\\tilde{v}\\\tilde{p}\end{Bmatrix} \tag{2b}$$

in which $\{\Phi\}$, $\{\Phi_0\}$, and $\{\tilde{\Phi}\}$ are vectors containing the total, steady, and acoustic variables in the flow, respectively. Further, viscous and heat conduction effects are considered important only on the mean process and an ideal gas is assumed to eliminate temperature from the system. Under these assumptions, Eq. (1) can be written as the following hyperbolic system governing the acoustic perturbation variables:

$$\frac{\partial\{\tilde{\Phi}\}}{\partial t}+[A]\frac{\partial\{\tilde{\Phi}\}}{\partial x}+[B]\frac{\partial\{\tilde{\Phi}\}}{\partial y}+\{\tilde{F}\}=\{0\} \tag{2c}$$

$$[A]=\begin{bmatrix}u&\rho&0&0\\0&u&0&1/\rho\\0&0&u&0\\0&\rho c^2&0&u\end{bmatrix},\quad [B]=\begin{bmatrix}v&0&\rho&0\\0&v&0&0\\0&0&v&1/\rho\\0&0&\rho c^2&v\end{bmatrix} \tag{2d}$$

$$\{\tilde{F}\}=[A]\frac{\partial\{\Phi_0\}}{\partial x}+[B]\frac{\partial\{\Phi_0\}}{\partial y}-\frac{1}{\rho}\{F_v\} \tag{2e}$$

$$\{F_v\}=\begin{bmatrix}0\\ \rho\left[\dfrac{\partial}{\partial x}\tau_{xx}+\dfrac{\partial}{\partial y}\tau_{xy}\right]_0\\ \rho\left[\dfrac{\partial}{\partial x}\tau_{xy}+\dfrac{\partial}{\partial y}\tau_{yy}\right]_0\\ \rho(\gamma-1)\left[\dfrac{\partial q_x}{\partial x}+\dfrac{\partial u_0}{\partial x}\tau_{xx}+\dfrac{\partial v_0}{\partial x}\tau_{xy}+\dfrac{\partial q_y}{\partial y}+\dfrac{\partial u_0}{\partial y}\tau_{xy}+\dfrac{\partial v_0}{\partial y}\tau_{yy}\right]_0\end{bmatrix} \tag{2f}$$

in which $c=\sqrt{\gamma p/\rho}$ is the local speed of sound. The components of the viscous stress tensor τ_{xx}, τ_{yy}, and τ_{xy} and the heat flux vector q_x and q_y are computed in the usual manner.[13] The subscript "0" outside each set of braces in Eq. (2f) is used to indicate that all viscous and heat conducting terms within are calculated using only the steady flow variables.

Equation (2c) is the set of hyperbolic nonlinear equations governing acoustic disturbances in a viscous flowing fluid. Note that in the absence of any disturbance, Eq. (2c) reduces to the Navier-Stokes equations ($\{\tilde{F}\}=\{0\}$) that govern the viscous steady flow. No exact analytical solutions to the equations of acoustics for an arbitrary steady flow exist, and they have not yet yielded even to digital-computer solutions in this most general form. Before the system can be solved numerically, however, a set of boundary conditions consistent with Eq. (2c) must be specified.

Acoustic Boundary Conditions

Free boundaries, such as those at inflow and outflow, across which matter and information are free to pass, are quite difficult to analyze. These boundaries do not arise from a natural physical situation, and suitable forms of the boundary condition required on them are not obvious. It is necessary that some of these conditions model a specified physical situation, such as a termination or source of noise. Other conditions should be consistent with the compatibility equations of the acoustic system so that the problem will be well posed.

Recently, there has appeared a paper by Thompson[11] that is aimed at developing nonreflecting characteristic boundary conditions for Euler's equations. For the specific examples to which he applied the conditions, the time dependence of the field exists only over brief intervals. For these he found that his conditions indeed generated no reflections. On the other hand, for problems associated with inviscid acoustic waves, the unsteady behavior never disappears, and it will be seen that the conditions of Ref. 11 do not reproduce outgoing acoustic solutions except for planar waves. In this section, an analysis similar to that of Thompson will be used to derive boundary conditions for the acoustic system. In addition to being consistent with the compatibility equations of the acoustic system, the derived boundary conditions will be capable of modeling a specified physical situation.

Derivation of useful inflow-outflow boundary conditions for the acoustic system, Eq. (2c), is carried out here by writing the system in characteristic form. Here, the characteristic form of the equations is considered because the procedures for the application of boundary conditions are reasonably well defined and the characteristic method is applicable to any number of dimensions. Define the matrix $[S]$ whose rows contain the left eigenvectors of $[A]$:

$$[S]=\begin{bmatrix}1&0&0&-1/c^2\\0&0&1&0\\0&1/2c&0&1/2\rho c\\0&-1/2c&0&1/2\rho c\end{bmatrix} \tag{3a}$$

The matrix $[A]$ can be diagonalized by the similarity transformation:

$$[S][A][S]^{-1}=[\Lambda_S]$$

$$[\Lambda_S]=\begin{bmatrix}u&0&0&0\\0&u&0&0\\0&0&u+c&0\\0&0&0&u-c\end{bmatrix}\equiv\begin{bmatrix}\lambda_1&0&0&0\\0&\lambda_2&0&0\\0&0&\lambda_3&0\\0&0&0&\lambda_4\end{bmatrix} \tag{3b}$$

Here the diagonal elements of $[\Lambda_S]$ are the eigenvalues of $[A]$. To place Eq. (2c) in characteristic form, define the vector $\{F_S\}$ as

$$\{F_S\}=-[A]\frac{\partial\{\tilde{\Phi}\}}{\partial x} \tag{4}$$

Multiplying Eq. (4) by $[S]$ and using Eq. (3) gives

$$[S]\{F_S\}+[\Lambda_S][S]\frac{\partial\{\tilde{\Phi}\}}{\partial x}=\{0\} \tag{5}$$

The solution to Eq. (5) can be expressed in the form

$$\{F_S\} = \begin{Bmatrix} \dfrac{\lambda_3 b_{3S}}{2c^2} + \dfrac{\lambda_4 b_{4S}}{2c^2} - \dfrac{\lambda_1 b_{1S}}{c^2} \\ \dfrac{\lambda_3 b_{3S}}{2\rho c} - \dfrac{\lambda_4 b_{4S}}{2\rho c} \\ \lambda_2 b_{2S} \\ \dfrac{\lambda_3 b_{3S}}{2} + \dfrac{\lambda_4 b_{4S}}{2} \end{Bmatrix} \tag{6a}$$

where

$$b_{1S} = -\left[\frac{\partial \tilde{p}}{\partial x} - c^2 \frac{\partial \tilde{\rho}}{\partial x}\right], \quad b_{2S} = -\frac{\partial \tilde{v}}{\partial x} \tag{6b}$$

$$b_{3S} = -\left[\frac{\partial \tilde{p}}{\partial x} + \rho c \frac{\partial \tilde{u}}{\partial x}\right], \quad b_{4S} = -\left[\frac{\partial \tilde{p}}{\partial x} - \rho c \frac{\partial \tilde{u}}{\partial x}\right] \tag{6c}$$

In physical terms, b_{1S} and b_{2S} can be identified with entropy and vorticity waves, respectively, whereas b_{3S} and b_{4S} are associated with acoustic waves propagating in opposite directions. It is this identification of the disturbance field that enables physical boundary conditions to be determined.

In this paper, the total axial velocity component at the inflow and outflow boundaries is considered to be subsonic so that $u_0 + \tilde{u} \le c$. In addition, it is assumed that $|\tilde{u}| < u_0$, because $\tilde{u}$ is meant to be a disturbance to the steady flow. Therefore, the sign of the characteristic velocities in Eq. (6a) can be determined at the inflow and outflow boundary. Note that the characteristic velocities λ_1, λ_2, and λ_3 are all positive whereas λ_4 is negative. It follows therefore that the characteristics associated with λ_1, λ_2, and λ_3 have positive slope at the inflow boundary and are entering the domain whereas the characteristic associated with λ_4 has a negative slope at inflow and is leaving the domain. Additionally, characteristics that enter the inflow boundary leave the outflow boundary and vice versa. It is therefore convenient to split $\{F_S\}$ into contributions from the positive and negative slope characteristics in the form

$$\{F_S\} = \{F_S^+\} + \{F_S^-\} \tag{7a}$$

where

$$\{F_S^+\} = \begin{Bmatrix} \dfrac{\lambda_3 b_{3S}}{2c^2} - \dfrac{\lambda_1 b_{1S}}{c^2} \\ \dfrac{\lambda_3 b_{3S}}{2\rho c} \\ \lambda_2 b_{2S} \\ \dfrac{\lambda_3 b_{3S}}{2} \end{Bmatrix} \tag{7b}$$

$$\{F_S^-\} = \frac{1}{2}\begin{Bmatrix} \dfrac{\lambda_4 b_{4S}}{c^2} \\ -\dfrac{\lambda_4 b_{4S}}{\rho c} \\ 0 \\ \lambda_4 b_{4S} \end{Bmatrix} \tag{7c}$$

Here, $\{F_S^+\}$ represents wave components propagating in the positive x direction, and $\{F_S^-\}$ denotes wave components propagating in the negative x direction. Substituting Eqs. (7) into Eq. (2c) gives

$$\frac{\partial\{\tilde{\Phi}\}}{\partial t} = \{F_S^+\} + \{F_S^-\} - \{F\}, \quad \{F\} = [B]\left\{\frac{\partial \tilde{\Phi}}{\partial y}\right\} + \{\tilde{F}\} \tag{8}$$

Equation (8) expresses the temporal derivatives of the acoustic system at the inflow or outflow boundary in terms of the characteristic velocities λ_1, λ_2, λ_3, and λ_4. To develop boundary conditions for Eq. (2c), it is fruitful to work with Eq. (8), which will be referred to as the characteristic form of Eq. (2c).

The characteristic theory[13] indicates that either the incoming wave field or at least a relationship between it and the outgoing wave field must be specified at the boundaries using known data. Therefore it is necessary to specify the quantities b_{1S}, b_{2S}, and b_{3S} at the inflow boundary or at least to specify these unknowns in terms of the outgoing wave quantity b_{4S}. Thompson[11] suggests that, in hydrodynamics, a nonreflecting inflow boundary condition is obtained by annihilating the quantities composing the incoming wave field (i.e., $b_{1S} = b_{2S} = b_{3S} = 0$), giving $\{F_S^+\} = \{0\}$ at inflow. The results of the current work show that this does not eliminate reflections of acoustic waves at the inflow except in the planar case.

A correct specification of b_{1S}, b_{2S}, and b_{3S} at inflow remains one of the most interesting questions associated with acoustic problems. It is necessary that three physical conditions be prescribed at inflow to determine these quantities. Further, it is necessary to specify these quantities such that the appropriate physical situation of reflecting or nonreflecting inflow is modeled. The boundary conditions derived here are based on the following assumptions:

1) The quantity b_{1S} is specified as some known function $b(y)$ at inflow, so that

$$b_{1S} = -\left[\frac{\partial \tilde{p}}{\partial x} - c^2 \frac{\partial \tilde{\rho}}{\partial x}\right] = b(y) \tag{9a}$$

Note that c is the local and not the ambient sound speed. Although the entropy function $b(y)$ is usually assumed to be zero for linear acoustic disturbances, it may be quite difficult to determine for more practical problems.

2) The inflow vorticity of the disturbance ζ is known. This allows the quantity b_{2S} to be written in terms of the known inflow vorticity as

$$b_{2S} = -\left[\zeta + \frac{\partial \tilde{u}}{\partial y}\right], \quad \zeta = \frac{\partial \tilde{v}}{\partial x} - \frac{\partial \tilde{u}}{\partial y} \tag{9b}$$

3) By analogy with the procedure used with many mechanical systems, the ratio between the acoustic pressure and normal velocity at the boundary z is assumed known:

$$z = \frac{\tilde{p}(y,t)}{\tilde{u}(y,t)} \tag{9c}$$

The quantity z will be referred to here as the acoustic impedance, although it is not the complex quantity usually defined for steady-state harmonic motion. Differentiation of Eq. (9c) with respect to time gives

$$\frac{\partial \tilde{p}}{\partial t} - z\frac{\partial \tilde{u}}{\partial t} - \tilde{u}\frac{\partial z}{\partial t} = 0 \tag{9d}$$

Then substitution of the time derivatives of $\tilde{p}$ and $\tilde{u}$ into Eq. (9d) via the fourth and second components of Eq. (8) allows the quantity b_{3S} to be written in terms of the given inflow impedance in the form

$$b_{3S} = z_1[z_0 - z_2\lambda_4 b_{4S}] \tag{9e}$$

$$z_1 = \frac{1}{\lambda_3[1 - (z/\rho c)]}, \quad z_2 = \left(1 + \frac{z}{\rho c}\right) \tag{9f}$$

$$z_0 = 2\left(F_4 - zF_2 + \tilde{u}\,\frac{\partial z}{\partial t}\right) \tag{9g}$$

in which F_i denotes the ith component of $\{F\}$. One way of considering an inflow boundary condition of the form of Eqs. (9) is as a means of determining the incoming waves at inflow in terms of those that are outgoing:

$$\{F_S^+(x,y,t)\} = [D(x,y,t)]\{F_S^-(x,y,t)\} + \{E(x,y,t)\}$$

where the matrix $[D]$ and vector $\{E\}$ can be determined from Eqs. (9). In this vein, the current method does not represent a departure from other analyses.

The three inflow boundary conditions specified in Eqs. (9a), (9b), and (9e) have the advantage of allowing for the most free-flow adjustment at inflow but have the disadvantage of requiring that the three critical input parameters ζ, $b(y)$, and z are known. A significant effort may be required to obtain these three parameters, and depending upon their specification, the acoustic waves may or may not be reflected from the inflow boundary. Values of these three parameters that simulate an outgoing plane wave are easily determined since, in that case, all the y derivatives in $\{F\}$ vanish and the only way to eliminate the incoming information in $\{F_S^+\}$ is by annihilating b_{1S}, b_{2S}, and b_{3S}. The following value of the three inflow parameters will simulate such a condition:

$$b(y) = 0, \qquad \zeta = 0, \qquad z = -\rho c$$

Thus for plane waves, the conditions of the current work reduce to those of Thompson[11]:

$$b_{1S} = b_{2S} = b_{3S} = 0$$

Boundary condition treatment along the outflow boundary is very similar to that at inflow. Only one characteristic λ_4 enters the computational domain at outflow. Thus b_{4S} is specified at the outflow boundary in terms of the outgoing wave components and the known source pressure. The form that properly models the sound source is readily obtained by replacing $(\partial\tilde{p}/\partial t)$ with $(\partial p_s(y,t)/\partial t)$ in the fourth component of Eq. (8) and solving for the incoming wave component b_{4S}, which yields

$$b_{4S} = \frac{1}{\lambda_4}\,[z_3 - \lambda_3 b_{3S}] \tag{10a}$$

$$z_3 = 2\left(\frac{\partial p_s}{\partial t}(y,t) + F_4\right) \tag{10b}$$

The upper boundaries, $y = 0$ and $y = H$, are considered rigid and nonheat-conducting, and the proper physical boundary conditions in this case are

$$\tilde{v} = 0, \qquad \frac{\partial\tilde{p}}{\partial y} = 0, \qquad \frac{\partial\tilde{\rho}}{\partial y} = 0 \tag{11}$$

The axial velocity $\tilde{u}$ cannot be given along a rigid boundary or the initial-boundary-value problem will in general be overspecified.

While the procedure for the application of boundary conditions may appear to be straightforward, some words concerning their implementation are appropriate to avoid any confusion. The implementation of boundary conditions into various numerical schemes has been known to produce spurious oscillations that may destabilize the scheme. The boundary conditions are implemented here as follows:

1) Along the inflow boundary, Eq. (2c) is integrated numerically in time using the expressions for b_{1S}, b_{2S}, and b_{3S} given in Eqs. (9a), (9b), and (9e) but with b_{4S} given by Eq. (6c).

2) At outflow, Eq. (2c) is integrated numerically using the expression for b_{4S} given by Eq. (10a) with b_{1S}, b_{2S}, and b_{3S} given by Eqs. (6).

3) Along the solid walls, Eq. (11) is used to determine $\tilde{v}$, $\tilde{p}$, and $\tilde{\rho}$ whereas the second component of Eq. (2c) is integrated numerically along the wall to determine the axial velocity $\tilde{u}$. First-order one-sided differences are used to determine all derivatives normal to the solid boundary.

4) Finally, $\tilde{u}$ is determined at the inflow corners from the known vorticity condition and the rigid wall condition

$$\frac{\partial\tilde{u}}{\partial y} = -\zeta$$

Numerical Solution Technique

Having developed boundary conditions for the acoustic system, attention is now focused on a numerical solution technique so that the derived boundary conditions can be tested. One specific algorithm that has gained wide use and acceptance for solving time-dependent problems in fluid dynamics was developed by MacCormack.[12] To date, this algorithm has not been used to integrate the acoustic system. It is used here to integrate both the differential equations and boundary conditions for the acoustic system. It should be noted that the differential equations and boundary conditions are fully coupled, and both are integrated using MacCormack's algorithm. For more details on MacCormack's method in fluid dynamics, consult the excellent reference by Anderson.[13]

Results and Discussion

It is instructive to see how well the algorithm described in this paper works in practice. To do so, the solution to the equations of acoustics using MacCormack's algorithm have been vectorized and programmed on the VPS-32 computer (i.e., a 64-bit vectorizable computer). A series of model problems were chosen to verify the integrity of the numerical solution to the acoustic system and to test the effects of the derived boundary conditions. Dimensions of the channel are in meters with $L = H = 0.5$ (throughout this work, the MKS units of measure will be used). A 41×41 grid was specified with uniformly spaced increments in both coordinate directions. Results presented here are restricted either to uniform flow results for which exact steady-state solutions to the acoustic system are known or to parallel sheared flows whose steady-state solutions were calculated numerically using a modal method.[14] Results were calculated with an ambient base state (i.e., $p_0 = 101{,}325$ N/m^2, $c_0 = 343$ m/s, $\rho_0 = 1.2$ kg/m^3) with $v_0 = 0$. The sound source is considered to be a single mode oscillating sinusodially with a frequency of $1100/\pi$ Hz (i.e., $\omega = 2\pi f = 2200$).

Plane Wave Source in Uniform Flow

The first problem to be given consideration is that of a plane acoustic wave propagating through the domain in the presence of a uniform flow. The critical inflow parameters were chosen such that the acoustic waves will exit the channel without reflecting (i.e., $b(y) = 0$, $z = -\rho c$, $\zeta = 0$). The channel is assumed to be initially quiescent; that is, at $t = 0$ the acoustic variables are taken as zero everywhere except at the outflow boundary. Application of the noise source at outflow will drive the acoustic disturbance through the domain. An exact solution derived from linear theory is used to initialize the disturbance at outflow, and the sound source is defined as

$$p_s(y,t) = A_0^s \sin(\omega t), \qquad A_0^s = \epsilon p_0 \tag{12}$$

where the parameter ϵ must be considerably less than unity for the solution to exhibit linear properties. Finally, plane wave results are plotted only at the centerline (i.e., $y = H/2$), since calculated results did not show any dependence on y.

Graphs of the normalized acoustic pressure computed from MacCormack's method at the 50th and 100th time step are compared to the exact steady-state solution in Figs. 2a and 2b, respectively. Computations are for a Mach number of 0.1 and

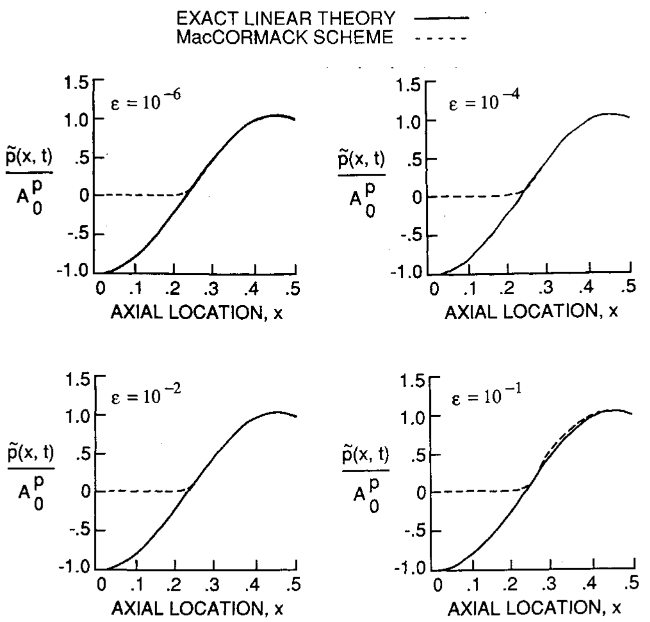

Fig. 2a Comparison of normalized steady-state acoustic pressure at the 50th time step for a plane wave source at 0.1 Mach number.

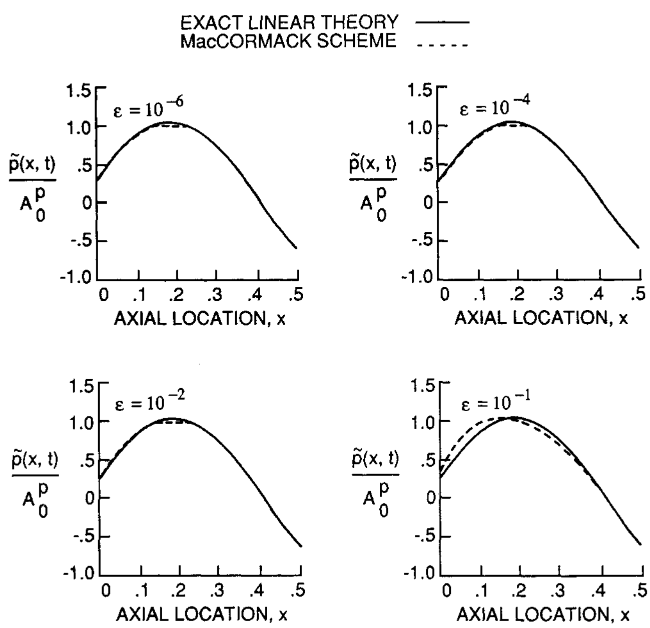

Fig. 2b Comparison of normalized steady-state acoustic pressure at the 100th time step for a plane wave source at 0.1 Mach number.

for values of ϵ shown. The solid curve denotes the exact steady-state solution from linear theory, whereas the broken line denotes the solution obtained from MacCormack's scheme. At the 50th time step, the initial transient has propagated over approximately 50% of the channel. Further, the initial transient has already reached the inflow boundary at the 100th time step. No evidence of any reflections as a result of the chosen inflow boundary condition appears in the figures, and the solution reaches the exact steady state at each axial location as soon as the initial transient has passed. Note that for even this low Mach number case, there are differences between the exact linear solution and the numerical solution for the largest value of source amplitude (i.e., $\epsilon = 10^{-1}$). Further, this difference appears to grow with distance from the sound source. This is evidence that nonlinear acoustic effects are starting to occur for the higher amplitude source, an effect that is not accounted for in linear theory. Note that the choice of $z = -\rho c$ gives no reflections for large values of ϵ.

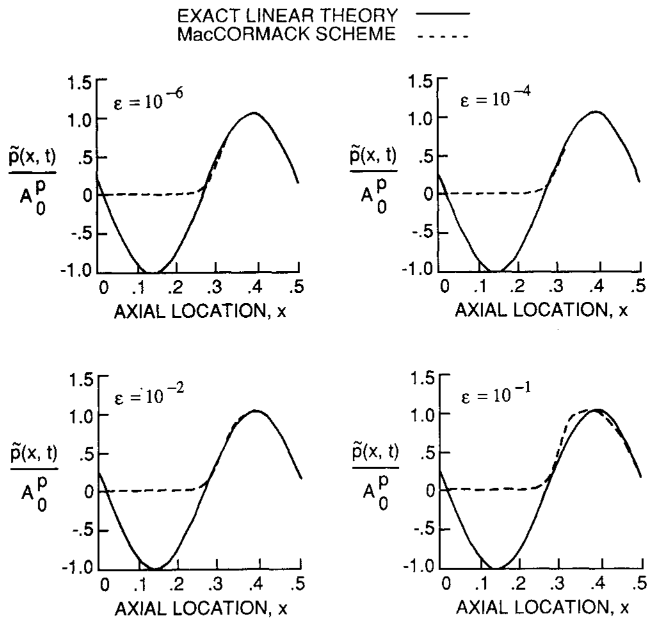

Fig. 3a Comparison of normalized steady-state acoustic pressure at the 100th time step for a plane wave source at 0.5 Mach number.

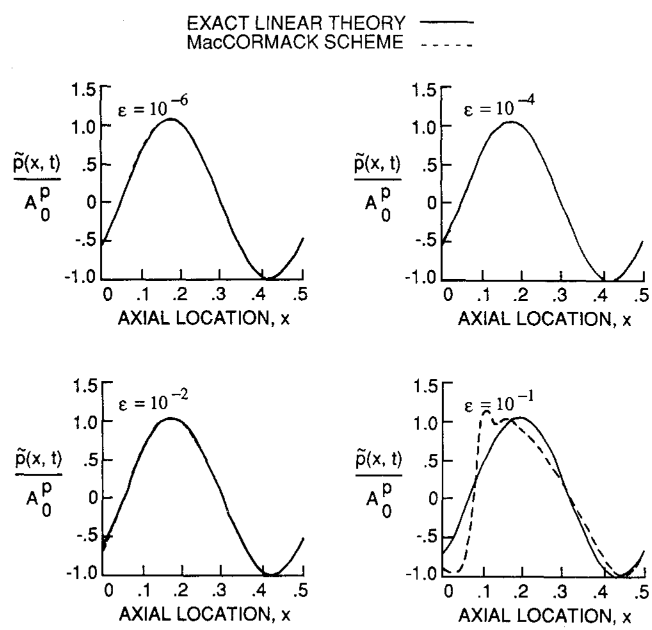

Fig. 3b Comparison of normalized steady-state acoustic pressure at the 400th time step for a plane wave source at 0.5 Mach number.

On the other hand, when the linearized expression $z = -\rho_0 c_0$ was used with the larger values of ϵ, unstable results were obtained.

As the mean flow Mach number increases, the acoustic wavelength is compressed by the flow. This is illustrated in Figs. 3a and 3b in which the Mach number was increased from 0.1 to 0.5. At the 100th time step, the initial transient has not reached the inflow boundary. At the 400th time step, the initial disturbance has already passed the inflow boundary, and no reflections are generated as a result of the inflow boundary conditions. Thus, the inflow boundary conditions are truly nonreflecting. When compared to the 0.1 Mach number case, these calculations show that the convected effects of the flow have shortened the acoustic wavelength considerably and have substantially increased the time required for the initial disturbance to exit the channel. Therefore, as Mach number increases, the computational cost for obtaining the steady-state solution using this algorithm will increase substantially. Note also that the numerical solution at axial location x reaches steady state as soon as the initial disturbance passes. Finally, the largest source amplitude results in significant nonlinear effects at steady state (Fig. 3b) and in the occurrence of an acoustic shock at approximately 0.085 m from the inflow boundary.

Higher-Order Mode in Uniform Flow

The second problem to be considered is that of a higher-order mode propagating down the channel. This case is of particular interest since transverse acoustic gradients will be present in the acoustic solution, although not in the mean flow. Again, the channel is initially quiescent, and the initial condition at outflow is initialized with an exact solution derived from linear theory. The sound source is the first-order hardwall mode:

$$p_s(y,t) = A_1^p \cos(\pi y/H) \sin(\omega t), \qquad A_1^p = \epsilon p_0 \tag{13}$$

The inflow boundary condition should simulate a reflection-free condition, since this is the basis for the exact solution to which the numerical solution will be compared. At the outset of this work, Thompson's conditions were tried with the current numerical scheme on a number of model problems

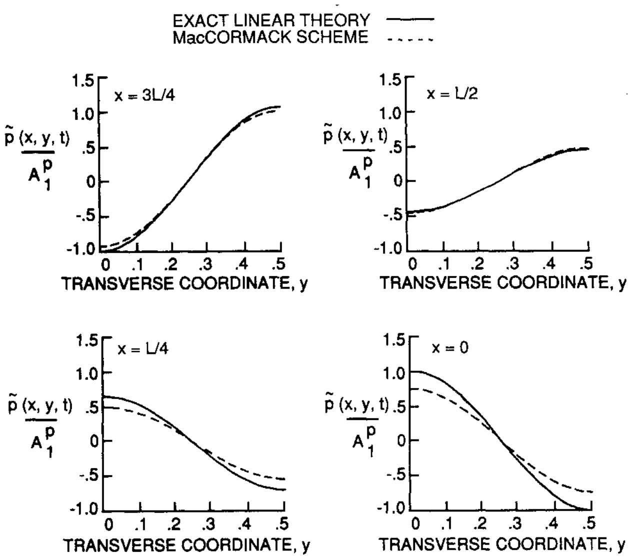

Fig. 4a Comparison of normalized acoustic pressure at the 400th time step for a nonplanar source at 0.5 Mach number.

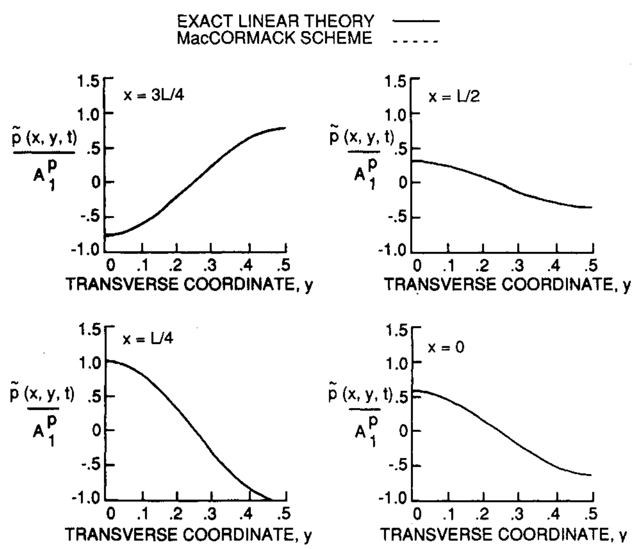

Fig. 4b Comparison of normalized acoustic pressure at the 1000th time step for a nonplanar source at 0.5 Mach number.

in which the sound source was nonplanar. In each case, these so-called nonreflecting boundary conditions led to reflections and eventually to unstable results for each problem attempted. To simulate a reflection-free inflow, it was decided that the following expressions derived from linearized theory should be used in Eqs. (9a–9g):

$$z = -\frac{(\omega + K_1 u_0)\rho_0}{K_1} \tag{14a}$$

$$\zeta = 0 \tag{14b}$$

$$b(y) = 0 \tag{14c}$$

where K_1 is the axial propagation constant obtained from linear theory. Note that this choice for the three inflow parameters does not annihilate the quantities b_{2S} and b_{3S}.

Plots of the acoustic pressure distribution at various axial stations for a uniform, 0.5 Mach number flow are shown in Figs. 4a and 4b. The distribution across the channel has been plotted with $\epsilon = 10^{-4}$ so that a comparison with linear theory can be made. Two important points should be observed. First, although approximately 250 time steps are required for the initial disturbance to reach the inflow boundary, the steady-state solution has not been reached after 400 time steps (Fig. 4a). Second, the numerical solution and the exact steady-state solution are nearly identical at the 1000th time step (Fig. 4b) with no evidence of any reflections as a result of the chosen inflow boundary conditions. It should be noted that the algorithm gave unstable results for $\epsilon = 10^{-1}$. This was simply another indication that use of linearized expressions such as Eqs. (14a–14c) may lead to invalid results for nonlinear acoustic propagation.

Higher-Order Mode in a Sheared Flow

The third and final problem considered is that of a higher-order mode propagating through a sheared mean flow. Unlike the two previous examples, this problem does not have exact analytical steady-state solutions, and even noflow linear results are difficult to obtain.[14] An approximate solution, based upon the sheared flow modes, has been used as a basis for comparison. The mean flow Mach number profile is a noslip profile of the form

$$M(y) = M_0 \sin(\pi y/H)$$

where M_0 is the centerline Mach number chosen as 0.5 for results presented here. Note that this mean flow profile is an exact solution to the steady form of Eq. (1). Again, the channel is assumed initially quiescent except at outflow, where the modal solution for the second-order mode is employed. The sound source is expressed in the form

$$p_s(y,t) = A_1^p P_1(y) \sin(\omega t), \qquad A_1^p = \epsilon p_0$$

where $P_1(y)$ is the second sheared flow mode obtained from linear theory and $\epsilon = 10^{-4}$. To simulate a reflection-free inflow, the three critical inflow parameters $b(y)$, ζ, and z are

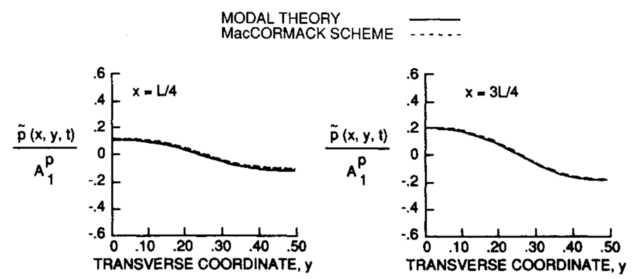

Fig. 5 Comparison of normalized acoustic pressure for a sheared mean flow at the 1500th time step.

determined from a solution technique similar to that of Ref. 14 for the second-order mode.

Figure 5 compares the MacCormack and modal solution at the 1500th time step. Excellent agreement between the two approximate solutions has been obtained. It should be noted that it takes approximately 500 time steps before the disturbance reaches the termination and that 1000 additional time steps are needed before the solution reaches steady state. It is clear, however, that the derived inflow-outflow boundary conditions give good results in a sheared mean flow and that no instabilities arise.

Conclusions

Based upon the results of this paper, the following conclusions have been drawn:

1) The full two-dimensional time-dependent nonlinear acoustic system requires three physical boundary conditions on an inflow boundary and one physical boundary condition on the outflow boundary.

2) When generalized to acoustic systems, the conditions in Ref. 11 give unstable results for nonplanar acoustic sources.

3) The most natural choice for the three inflow boundary conditions appears to be a specification of the normal acoustic impedance and a pressure density relationship, along with the acoustic vorticity. On the other hand, a specification of the acoustic pressure on the outflow boundary was shown to give stable and consistent results.

4) The explicit predictor-corrector method of MacCormack represents a stable numerical method for integration of the full time-dependent nonlinear acoustic system interior to a domain coupled with the boundary conditions.

Although the boundary conditions employed here were tested only on model problems for which exact solutions were known, they should be applicable to more complicated problems, provided the three critical inflow parameters can be determined. In many cases of practical interest, these parameters could be measured or approximated, particularly when steady-state solutions are sought.

Acknowledgments

The work of the second author was funded by NASA under Cooperative Agreement NCC 1-14. The authors wish to express their appreciation to M. Yousuff Hussaini for his technical support during the course of this research.

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